§ 11. Chern-Simons functionals and Ray-Singer torsion

Let M be a compact oriented 3-manifold without boundary. Set $G = SU(2)$

$\longrightarrow$ consider principal G bundle P over M

$\mathcal{A}$ = space of connections

$\cong \Omega(M, \mathfrak{g})$

$CS : \mathcal{A} \longrightarrow \mathbb{R}$ is defined by

$$CS(A) = \frac{1}{8\pi^2} \int_M Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right), \quad A \in \mathcal{A}$$

We have for $g \in \mathcal{G} = Map(M, G)$:

$$g^* A = g^{-1} A g + g^{-1} d g$$

and $CS(g^* A) = CS(A) + n, \quad n \in \mathbb{Z}$

Note that

$$g^* A = A + t\underbrace{(df + [A, f])}_{=: d_A f} + \mathcal{O}(t^2), \quad A \in \mathcal{A}.$$

Let $\alpha \in \mathcal{A}$ be a flat connection and denote by Ω^j_α the space of j-forms on M with values in $\mathfrak{g}_\alpha$

$\rightarrow$ de Rham complex:

$$0 \to \Omega^0_\alpha \to \Omega^1_\alpha \to \Omega^2_\alpha \to \Omega^3_\alpha \to 0$$

with differential d_α. ($d_\alpha \circ d_\alpha = 0$ follows from flatness)

In the following, we assume: $H^*(M, g_\alpha) = 0$

<u>Assymptotic expansions:</u>

recall: $\int_{-\infty}^{\infty} e^{-\mu x^2} dx = \sqrt{\frac{\pi}{\mu}}$, $\mu > 0$

By analytic continuation:

$$\int_{-\infty}^{\infty} e^{i\lambda x^2} dx = \sqrt{\frac{\pi}{|\lambda|}}\, e^{\frac{\pi i}{4}\frac{\lambda}{|\lambda|}} \quad , \lambda \in \mathbb{R}, \lambda \neq 0 \qquad (*)$$

Let Q be a non-degenerate quadratic form in $x_1, \dots, x_n$ and denote by $\operatorname{sgn} Q$ its signature. Then

$$\int_{\mathbb{R}^n} e^{iQ(x_1, \dots, x_n)} dx_1 \dots dx_n = \frac{\pi^{n/2}}{\sqrt{|\det Q|}}\, e^{\frac{\pi i}{4} \operatorname{sgn} Q}$$

Consider now the asymptotics of the integral

$$\int_{\mathbb{R}^n} e^{ikf(x_1, \dots, x_n)} dx_1 \dots dx_n$$

as $\kappa \to \infty$. Let's deal with one variable first:

$$g(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa f(x)}\, dx \quad , \quad \kappa \to \infty$$

"Stationary phase method":
major contribution to above integral arises from critical point of f:

$$g(\kappa) \sim \int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{i\kappa f(x)}\, dx = \int_{u_1}^{u_2} e^{i\kappa(f(x_0)+u^2)} \frac{2u}{f'(x)}\, du$$

where $u^2 = f(x) - f(x_0)$

$$\left[\frac{du}{dx} = \frac{1}{2} \frac{f'(x)}{u} \right.$$

By using

$$\lim_{x\to x_0} \frac{2u}{f'(x)} = \lim_{x\to x_0} \frac{2\left(f(x)-f(x_0)\right)^{1/2}}{f'(x)}$$

$$= \lim_{x\to x_0} 2 \frac{\left[\left(f(x)-f(x_0)\right)^{1/2}\right]'}{f''(x)}$$

$$= \lim_{x\to x_0} 2 \, \frac{1}{2} \frac{f'(x)}{\left(f(x)-f(x_0)\right)^{1/2}} \frac{1}{f''(x)}$$

$$\Longleftrightarrow \left(\lim_{x\to x_0} \frac{2u}{f'(x)}\right)^2 = 2 \lim_{x\to x_0} \frac{1}{f''(x)} = \frac{2}{f''(x_0)}$$

$$\Longrightarrow \quad \lim_{x\to x_0} \frac{2u}{f'(x)} = \left(\frac{2}{f''(x_0)}\right)^{1/2}$$

we obtain

$$g(\kappa) \sim \left(\frac{2}{f''(x_0)}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\kappa u^2 + i\kappa f(x_0)}\, du$$

$$\overset{(*)}{=} \left(\frac{2\pi}{\kappa f''(x_0)}\right)^{1/2} e^{i\kappa f(x_0) + \frac{\pi i}{4}} \qquad \text{for } \kappa \to \infty$$

Similarly, we obtain

$$\int_{\mathbb{R}^n} e^{i\kappa f(x_1, \dots, x_n)}\, dx_1 \cdots dx_n$$

$$\sim_{\kappa \to \infty} \sum_{\alpha} \frac{\pi^{n/2} e^{i\kappa f(\alpha)}}{\kappa^{n/2} \sqrt{|\det H(f,\alpha)|}}\, e^{\frac{\pi i}{4} \operatorname{sgn} H(f,\alpha)}$$

where the sum is now over all critical points α of f and $H(f,\alpha)$ is the Hessian of f at α.

Let us now go back to the Chern-Simons functional

$\longrightarrow$ critical points = flat connections

Denote by $(\Omega^*_\alpha, d_\alpha)$ the de Rham complex associated with α a flat connection. We have

$$T_\alpha \mathcal{A} \cong \Omega^1(M, \mathfrak{g})$$

with inner product

$$\langle A, B \rangle = -\frac{1}{8\pi^2} \int_M \mathrm{Tr}(A \wedge * B)$$

$\longrightarrow$ obtain orthogonal decomposition with respect to $\langle , \rangle$:

$$T_\alpha \mathcal{A} = \mathrm{Im}\, d_\alpha \oplus \mathrm{Ker}\, d_\alpha^*$$

Here $d_\alpha : \Omega^0_\alpha \longrightarrow \Omega^1_\alpha$, $d_\alpha^* : \Omega^1_\alpha \longrightarrow \Omega^0$ adjoint operator

$\ulcorner$ Write $C = A + \underbrace{d_\alpha B}_{\neq 0} \in T_\alpha \mathcal{A}$

Then $0 = \langle A, \underbrace{d_\alpha^2 B}_{=0} \rangle = \langle d_\alpha^* A, B \rangle$

$\llcorner$ as B arbitrary $\longrightarrow A \in \mathrm{Ker}\, d_\alpha^*$

The subspace $\mathrm{Im}\, d_\alpha \subset T_\alpha \mathcal{A}$ is the "gauge orbit" $\mathcal{G}_\alpha$ at α.

Let us compute the Hessian of CS at α:

Set $\alpha_t = \alpha + t\beta$

$$\longrightarrow CS(\alpha_t) = CS(\alpha) + \frac{t^2}{8\pi^2} \int_M \mathrm{Tr}(\beta \wedge d_\alpha \beta) + \mathcal{O}(t^3).$$

$\longrightarrow$ Hessian: $Q(\beta) = \langle \beta, * d_\alpha \beta \rangle$

As CS is inv. under inf. gauge trfs.,
Q degenerates on $\mathscr{G}_\alpha$.

$\longrightarrow$ non-degenerate on $\Omega^1_\alpha / d_\alpha \Omega^0_\alpha$

<u>Computation of det Q:</u>

consider operator

$$P = \varepsilon(d_\alpha * + * d_\alpha)$$

acting on

$$\Omega^1_\alpha \oplus \Omega^3_\alpha \cong \Omega^1_\alpha \oplus \Omega^0_\alpha \quad \text{(Poincaré duality)}$$

where $\varepsilon|_{\Omega^0_\alpha} = 1 \qquad \varepsilon|_{\Omega^1_\alpha} = -1$

The action of P on the direct sum

$$\Omega^0_\alpha \oplus \operatorname{Im} d_\alpha \oplus \ker d_\alpha^*$$

is given by

$$P = \begin{pmatrix} 0 & -d_\alpha^* & 0 \\ -d_\alpha & 0 & 0 \\ 0 & 0 & Q \end{pmatrix}, \quad Q = - * d_\alpha$$

and we have $P^2 = \Delta^0_\alpha \oplus \Delta^1_\alpha$

Here $\Delta^j_\alpha = d_\alpha^* d_\alpha + d_\alpha d_\alpha^*$ for $j = 0, 1$.

"regularized determinant" of Laplace-op. Δ:

Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots$ be positive eigenvalues of Δ counted with multiplicities

$\rightarrow$ define "spectral zeta function":

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s}$$

$\rightarrow$ analytic continuation gives meromorphic function on $\mathbb{C}$ with definite values $\zeta_\Delta(0)$ and $\zeta_\Delta'(0)$

set $\det \Delta = \exp(-\zeta_\Delta'(0))$

$$\zeta_\Delta'(s) = \frac{d}{ds} \sum_{j=1}^{\infty} \exp(-\log\lambda_j \cdot s)$$

$$= \sum_{j=1}^{\infty} -\log\lambda_j \exp(-\log\lambda_j s)$$

$$\rightarrow \exp(-\zeta_\Delta'(0)) = \prod_{j=1}^{\infty} e^{\log\lambda_j} = \prod_{j=1}^{\infty} \lambda_j$$

It was shown by Ray and Singer that for a flat connection α, that

$$T_\alpha(M) = \frac{(\det \Delta^0_\alpha)^{3/2}}{(\det \Delta^1_\alpha)^{1/2}}$$

is a topological invariant of M

$\longrightarrow$ "Ray - Singer torsion"

<u>Proposition 1:</u>

The regularized determinant of Hessian Q of CS-functional at flat con. α is

$$\frac{\sqrt{\det d_\alpha^* d_\alpha}}{\sqrt{|\det Q|}} = T_\alpha^{1/2}$$